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HOMOGENEOUS AUTONOMOUS SYSTEMS WITH THREE INDEPENDENT VARIABLES†

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Double-wave solutions of equations with three independent variables are studied. The case when a homogeneous autonomous system consisting of four independent quasilinear first-order differential equations can be formed to study compatibility is considered. All such systems having solutions with an arbitrary function that cannot be reduced to invariant ones are given and their solutions are found.

The classification of solutions with degenerate hodograph is one of the fundamental problems of the theory of r-fold waves. From the point of view of the group properties of differential equations an r-fold wave is a partially invariant solution (PIS) with respect to a subgroup of the transformation group G^{n+1} with Lie algebra L^{n+1} whose operator basis is [1]

$$\xi_0\partial = x_i\partial_i, \quad \xi_i\partial = \partial_i \quad (i=1,2,\dots,n)$$

where n is the number of independent variables.

PISs that cannot be reduced to invariant ones are a special case. This is connected with the fact that the problem of constructing invariant solutions is much simpler than that of constructing PISs. Namely, apart from a similitude transformation, two kinds of wave parameters $(\lambda^1, \dots, \lambda^r)$ are possible for an invariant r-fold wave [1].

A similitude transformation is defined by a linear transformation $x' = Vx$ of the independent variables with a non-singular $n\times n$ square matrix V. Besides, a PIS involves a more complex analysis of the compatibility of the resulting overdetermined systems than an invariant solution. It is therefore useful to elucidate in advance the form of irreducible multiple waves. In the literature there are only separate sufficient conditions for reducibility. For double waves Ovsyannikov's theorem [1] on the reduction to invariant solutions plays a fundamental role. In particular, if $n=3$ and there are five independent autonomous first-order homogeneous quasilinear equations satisfied by the wave parameters, then such solutions can be reduced to invariant ones.

Another key point in the study of solutions with degenerate hodograph is the investigation of the compatibility of overdetermined systems. Since the general analysis of the systems that arise presents difficulties, it has been carried out with additional assumptions regarding the solutions. Initially, these were kinematic and geometric conditions: it was assumed that the flow is a potential one or that the level curves are rectilinear [2–4]. We note that the requirement that the level curves should be rectilinear corresponds to a smaller invariance defect. Restrictions were also constructed on the basis of the algebraic structure of the system connected with so-called simple integral elements [5, 6]. Since in each case one must analyse the compatibility of an overdetermined system, it is more natural from the viewpoint of compatibility theory to classify the solutions with degenerate hodograph depending on the presence of an arbitrary function in the general solution.

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The study of solutions with an arbitrary function is based on the fact that every compatible system of differential equations is involutory after a finite number of extensions. If the system of differential equations is involutory, then the arbitrary function in the solution is determined by the Cartan characters, which are related in a certain way to the leading parametric derivatives. Thus, for a solution with an arbitrary function to exist it is necessary that the rank of the coefficient matrix multiplying the leading derivatives should be different from the total number of leading derivatives (for any extension).

In the case of three independent variables the classification of double waves almost always involves the study of systems consisting of four first-order homogeneous quasi-linear equations

$$\sum_{\alpha=1}^{3}(\lambda_\alpha p_j^\alpha(\lambda,\mu)+\mu_\alpha q_j^\alpha(\lambda,\mu))=0 \qquad (j=1,2,3,4) \tag{0.1}$$

Such systems and their solutions are studied below: all systems (0.1) having solutions with an arbitrary function that cannot be reduced to invariant ones are given and their general solutions are found. The present paper extends the results on the classification of multiple waves for the equations of gas dynamics and the theory of plasticity obtained in [7–10].

1. EQUIVALENCE TRANSFORMATION

Let $u=(\lambda,\ \mu)$ be the parameters of a double wave and let $\lambda_i=\partial\lambda/\partial x_i$, $\mu_i=\partial\mu/\partial x_i$ and $u_i=(\lambda_i,\ \mu_i)$ $(i=1,2,3)$. For system (0.1) the property of being homogeneous and autonomous is invariant under the following equivalence transformations:

(a) the choice of wave parameters $\lambda'=L(\lambda,\ \mu)$ and $\mu'=M(\lambda,\ \mu)$;

(b) a non-singular linear transformation of the independent variables.

By virtue of the double wave condition $\operatorname{rank}\partial(\lambda,\ \mu)/\partial(x_1,\ x_2,\ x_3)=2$ it can be shown by means of equivalence transformation that any system (0.1) of four independent equations can be reduced to one of the following two forms

$$\lambda_1=0,\quad \lambda_2=0,\quad \mu_3=0$$
$$\lambda_3+a(\lambda,\mu)\mu_1+b(\lambda,\mu)\mu_2=0 \qquad (a^2+b^2\neq 0) \tag{1.1}$$

or

$$u_3=Au_1,\qquad u_2=Bu_1 \tag{1.2}$$

where $A=(a_{ij}(\lambda,\ \mu))$ and $B=(b_{ij}(\lambda,\ \mu))$ are (2×2) square matrices.

2. THE SOLUTION OF SYSTEM (1.1)

On differentiating the last equation in (1.1) with respect to x_3 we obtain the equations (differentiation with respect to x_3 is denoted by a prime)

$$\lambda''+(a_\lambda\mu_1+b_\lambda\mu_2)\lambda'=0 \tag{2.1}$$
$$\lambda'''+(a_\lambda\mu_1+b_\lambda\mu_2)\lambda''+(a_{\lambda\lambda}\mu_1+b_{\lambda\lambda}\mu_2)\lambda'=0$$

Equations (1.1) and (2.1) imply the relationships

$$(b_\lambda\lambda'^2-b\lambda'')+\Delta_1\lambda'\mu_1=0,\qquad -(a_\lambda\lambda'^2-a\lambda'')+\Delta_1\lambda'\mu_2=0 \tag{2.2}$$
$$(\Delta_1=ab_\lambda-ba_\lambda)$$

If $\Delta_1 \neq 0$, then expressions for μ_1 and μ_2 can be found from Eqs (2.2). On substituting these expressions into (2.1), one can determine λ'''. Thus in the present case system (1.1) can have at most an arbitrary constant. It must therefore be assumed that $\Delta_1 = 0$. Then $b = g(\mu)a$ (without loss of generality it is assumed that $a \neq 0$) and

$$D_i(a_\lambda \lambda'^2 - a\lambda'') = \mu_i(a_{\lambda\mu}\lambda'^2 - a_\mu \lambda'') = 0 \quad (i = 1, 2)$$

(here and henceforth D_i is the total derivative with respect to x_i). Hence $aa_{\lambda\mu} - a_\lambda a_\mu = 0$ or $a = \phi\psi$ with some functions $\phi = \phi(\lambda)$ and $\psi = \psi(\mu)$. By the equivalence transformation with $dL = \phi^{-1}d\lambda$ and $dM = \psi d\mu$, these functions can be reduced to $\phi = \psi = 1$. Then (1.1) becomes a decoupled system. We get $\lambda = cx_3$, where $c \neq 0$ is an arbitrary constant, which, by the equivalence transformation is unimportant. For μ we have the equation $\mu_1 + g(\mu)\mu_2 = -c$, which can be integrated in the standard manner [11].

Theorem 1. All systems (1.1) having solutions with an arbitrary function are equivalent to the system

$$\lambda = x_3, \quad \mu_1 + g(\mu)\mu_2 = -1$$

3. THE SOLUTION OF SYSTEM (1.2)

First we note that

$$D_2(u_3 - Au_1) - D_3(u_2 - bu_1) \equiv Gu_{11} - C\langle u_1, u_1\rangle = 0 \tag{3.1}$$

Here $G = AB - BA$ with elements

$$g_{11} = -g_{22} = a_{12}b_{21} - a_{21}b_{12}$$
$$g_{12} = a_{12}(b_{22} - b_{11}) - b_{12}(a_{22} - a_{11}), \quad g_{21} = -a_{21}(b_{22} - b_{11}) + b_{21}(a_{22} - a_{11})$$

C is a bilinear mapping, whose coordinates are determined by A and B and their derivatives with respect to λ and μ.

If $\det G \neq 0$ then the solution of system (1.2) can have at most an arbitrary constant. Since systems having solutions with an arbitrary function are considered, it must be assumed that $\det G = 0$, i.e.

$$\begin{gathered} a_{12}a_{21}(b_{22} - b_{11})^2 - (a_{12}b_{21} + a_{21}b_{12})(b_{22} - b_{11})(a_{22} - a_{11}) + b_{12}b_{21}(a_{22} - a_{11})^2 - \Delta^2 = 0 \\ (\Delta = a_{12}a_{21} - b_{12}b_{21}) \end{gathered} \tag{3.2}$$

If $a_{12}a_{21}\Delta \neq 0$, then (3.2) implies that $(a_{22} - a_{11})^2 + 4a_{12}a_{21} \geqslant 0$. Under this condition A has real eigenvalues. This case is studied separately below.

If $a_{12}a_{21}\Delta = 0$, then either $a_{12}a_{21} = 0$, and so A also has real eigenvalues, or

$$\Delta = 0, \quad a_{12}a_{21} \neq 0 \tag{3.3}$$

In the latter case $G = 0$. But then (3.1) contains two homogeneous quadratic forms with respect to $u_1 = (\lambda_1, \mu_1)$

$$C\langle u_1, u_1\rangle = 0 \tag{3.4}$$

If at least one of the coefficients of the quadratic forms (3.4) is non-zero, it gives rise to a fifth first-order homogeneous autonomous quasilinear equation and, by the reduction theorem [1], such solutions can be reduced to invariant ones. In this case one must therefore assume that $C = 0$. Since $a_{12}a_{21} \neq 0$ by assumption, (3.2)–(3.4) yield the relationships

$$
\begin{gathered}
b_{21} = \frac{a_{21}b_{12}}{a_{12}}, \quad b_{22} = \xi + b_{11} \quad \left(\xi \equiv \frac{a_{22} - a_{11}}{a_{12}}\right) \\
\frac{\partial b_{12}}{\partial \lambda} - \frac{\partial b_{11}}{\partial \mu} + \frac{b_{12}}{a_{12}}\left(\frac{\partial a_{11}}{\partial \mu} - \frac{\partial a_{12}}{\partial \lambda}\right) = 0 \\
\frac{\partial b_{11}}{\partial \lambda} - \frac{a_{21}}{a_{12}}\frac{\partial b_{12}}{\partial \mu} + \xi\frac{\partial b_{11}}{\partial \mu} + \frac{b_{12}}{a_{12}}\left(\frac{a_{21}}{a_{12}}\frac{\partial a_{12}}{\partial \mu} - \frac{\partial a_{11}}{\partial \lambda} - \xi\frac{\partial a_{11}}{\partial \mu}\right) = 0
\end{gathered}
\tag{3.5}
$$

Under these conditions, system (1.2) is involutory with two arbitrary functions of one argument in the solution.

Theorem 2. Apart from equivalence transformations, system (1.2) has solutions with an arbitrary function that cannot be reduced to invariant ones only if A has real eigenvalues or conditions (3.5) are satisfied.

Below we consider the case when A has real eigenvalues.

Remark. The classification of double waves considered in all the papers known to us can be reduced to analysing the solutions of system (1.2) when A has real eigenvalues. In many of these publications this property is not pointed out explicitly. It is a consequence of the following. The classification of double waves involves transferring to the hodograph space $x_1 = P(\lambda, \mu, x_3)$, $x_2 = Q(\lambda, \mu, x_3)$, followed by obtaining a second-order degenerate algebraic equation in $\partial P/\partial x_3$ and $\partial Q/\partial x_3$, which splits into the product of two linear forms. It can be shown that this is only possible if A has real eigenvalues.

Now, let A have real eigenvalues. Without loss of generality (by the equivalence transformations) A can be assumed to be a Jordan matrix. Here it is necessary to distinguish between the two possibilities when A has a triangular or diagonal form. Different cases must be studied depending on the value of rank G.

We initially consider the case when $\operatorname{rank} G = 0$, i.e. $G = 0$. If $a_{12} = 1$, then $a_{22} = a_{11}$ and (3.5) can be obtained from $C = 0$. For $a_{12} = 0$ and for solutions irreducible to invariant ones it follows that $a_{21} = b_{12} = b_{21} = 0$ and

$$
(a_{22} - a_{11})\frac{\partial b_{11}}{\partial \mu} - (b_{22} - b_{11})\frac{\partial a_{11}}{\partial \mu} = 0 \quad (a_{22} - a_{11})\frac{\partial b_{22}}{\partial \lambda} - (b_{22} - b_{11})\frac{\partial a_{22}}{\partial \lambda} = 0 \tag{3.6}
$$

Under these conditions system (1.2) is also involutory with two arbitrary functions of one argument in the solution.

Now, let $\operatorname{rank} G = 1$. If A has the triangular form ($a_{22} = a_{11}$, $a_{12} = 1$. $a_{21} = 0$), then

$$
G = \begin{Vmatrix} b_{21} & b_{22} - b_{11} \\ 0 & -b_{21} \end{Vmatrix}
$$

and so $b_{21} = 0$ and $b_{22} - b_{11} \neq 0$. Thus B can be reduced to diagonal form. But then, by means of an equivalence transformation, the study of this case can be reduced to the case when A is diagonal, which is considered below.

For a diagonal matrix A we get

$$
G = (a_{22} - a_{11})\begin{Vmatrix} 0 & -b_{12} \\ b_{21} & 0 \end{Vmatrix}
$$

Since $\operatorname{rank} G = 1$, it follows that $a_{22} - a_{11} \neq 0$ and $b_{12}b_{21} = 0$. Below it is assumed, without loss of generality, that $b_{12}=0$ and $b_{21}\neq 0$.

Then the first equation in (3.1) has the form

$$\lambda_1 b_{21}\frac{\partial a_{11}}{\partial\mu}+\mu_1\left(\frac{\partial a_{11}}{\partial\mu}(b_{22}-b_{11})-\frac{\partial b_{11}}{\partial\mu}\right](a_{22}-a_{11})=0$$

Hence, by irreducibility, we find that

$$a_{11}=a_{11}(\lambda),\quad b_{11}=b_{11}(\lambda)$$

Moreover, the second equation in (3.1) can be written as

$$\begin{aligned}&\lambda_{11}=a\lambda_1^2+b\lambda_1\mu_1\\&a=\eta^{-1}\left[b_{21}\frac{\partial a_{11}}{\partial\lambda}-(a_{22}-a_{11})\frac{\partial b_{21}}{\partial\lambda}\right],\quad \eta=b_{21}(a_{22}-a_{11})\\&b=\eta^{-1}\left[(b_{22}-b_{11})\frac{\partial a_{22}}{\partial\lambda}-b_{21}\frac{\partial a_{22}}{\partial\lambda}-(a_{22}-a_{11})\frac{\partial b_{22}}{\partial\lambda}\right]\end{aligned}\tag{3.7}$$

If $b\neq 0$, the relationships

$$\begin{aligned}&D_3(\lambda_{11}-a\lambda_1^2-b\lambda_1\mu_1)-D_1D_1(\lambda_3-a_{11}\lambda_1)=0\\&D_2(\lambda_{11}-a\lambda_1^2-b\lambda_1\mu_1)-D_1D_1(\lambda_2-b_{11}\lambda_1)=0\end{aligned}\tag{3.8}$$

can be used to determine μ_{11}, which leads to solutions having arbitrary constants only. One must therefore assume that

$$b=0\tag{3.9}$$

All second-order derivatives are then eliminated in (3.9). They represent two homogeneous quadratic forms with respect to μ_1. From irreducibility and (3.8) we therefore get

$$a\frac{\partial a_{11}}{\partial\lambda}+\frac{\partial^2 a_{11}}{\partial\lambda^2}=0,\quad a\frac{\partial b_{11}}{\partial\lambda}+\frac{\partial^2 b_{11}}{\partial\lambda^2}=0,\quad \frac{\partial a}{\partial\mu}=0\tag{3.10}$$

Equations (3.9) and (3.10) ensure that system (2.1) is involutory with one arbitrary function of a single argument, and, in order to determine the solution, it is necessary to study the following three cases: (a) $a_{11}'\neq 0$, (b) $a_{11}'=0$, $b_{11}'\neq 0$, and (c) $a_{11}'=0$, $b_{11}'=0$.

If $a_{11}'\neq 0$, then we find from (3.10) that $b_{11}=c_1a_{11}+c_2$ with certain constants c_1 and c_2. After the linear transformation $x_1'=x_1+c_2x_2$, $x_2'=x_2$ and $x_3'=x_3+c_1x_2$ of the independent variables it follows that $b_{11}=0$ in system (1.2) written in the new system of coordinates.

By (3.10), we find from (1.2) and (3.7) that $a_{11}(\lambda)=-x_1/x_3$, apart from translations with respect to the independent variables. On applying the equivalence transformation $\lambda'=-a_{11}(\lambda)$ and $\mu'=\mu$, we determine from (3.10) that $a=0$ in the new variables (λ', μ'), which yields $\partial b_{21}/\partial\lambda\neq 0$ and $a_{22}=-\lambda-b_{21}(\partial b_{21}/\partial\lambda)^{-1}$. Here and below the prime is omitted. The integral (3.9) will be

$$b_{22}=(a_{22}+\lambda)(-\partial b_{21}/\partial\mu+b_{21}\psi(\mu))\tag{3.11}$$

where $\psi(\mu)$ is an arbitrary function. By the equivalence transformation $\lambda'=\lambda$ and $\mu'=f(\mu)$ with $f(\mu)$ satisfying the equation $f'+\psi f'^2=0$, one can assume that $\psi=0$ in (3.11). Reducing the remaining two equations in (1.2) to a homogeneous linear system by the standard method

[11], we obtain its solution $\Phi(\mu - b_{21}x_2/x_3,\ b_{21}/x_3) = 0$ with an arbitrary function $\Phi(\xi_1,\ \xi_2)$ $(\xi_1 = \mu - b_{21}x_2/x_3,\ \xi_2 = b_{21}/x_3)$. If $\Phi_{\xi_2} = 0$, then the solution is invariant. One must therefore assume that $\Phi_{\xi_2} \neq 0$. Finally, we observe that the resulting solution has a defect $\delta = 1$.

Cases (b) and (c) can be studied in a similar way. Here we will give just an outline. By the equivalence transformations one can assume that $a_{11} = 0$ in both these cases, $b_{11} = -\lambda$ in case (b), and $b_{11} = 0$ in case (c). It follows that $\lambda_{11} = 0$ in both cases. Since we also have $a = b_{21}^{-1}\partial b_{21}/\partial\lambda = 0$, it follows that $b_{21} = f(\mu)$. But then b_{12} becomes $b_{21} = 1$ under the transformation $\lambda' = \lambda$, $\mu' = \int f^{-1}(\mu)d\mu$. In order that there should be no reduction to invariant solutions, it is necessary that $a_{22} \neq 0$. If we set $\phi_\lambda = 1/a_{22}$, then Eq. (3.9) gives rise to expressions for b_{22}. Then, reducing the remaining two equations in system (1.2) to a homogeneous linear system, we can find its solution.

The following theorem is therefore true.

Theorem 3. Let A in system (1.2) have real eigenvalues. Then systems of the form (1.2) having solutions with an arbitrary function that are irreducible to invariant ones are equivalent to one of the following systems

(a) with coefficients $(b_{21}(\partial b_{21}/\partial\lambda) \neq 0)$

$$a_{11} = -\lambda, \quad b_{11} = 0, \quad a_{22} = -\lambda - b_{21}\left(\frac{\partial b_{21}}{\partial\lambda}\right)^{-1}, \quad b_{22} = b_{21}\frac{\partial b_{21}}{\partial\mu}\left(\frac{\partial b_{21}}{\partial\lambda}\right)^{-1}$$

and the general solution

$$\lambda = x_1/x_3, \quad \Phi(\mu - b_{21}x_2/x_3, b_{21}/x_3) = 0$$

(b) with coefficients

$$a_{11} = 0, \quad b_{11} = -\lambda, \quad a_{22} = 1/\phi_\lambda, \quad b_{21} = 1, \quad b_{22} = -\lambda + (\phi_\mu + \psi' e^{-\mu})/\phi_\lambda$$

and the general solution

$$\lambda = x_1/x_2, \quad \Phi((x_3/x_2 + \phi)e^{\mu} + \psi, \quad x_2 e^{-\mu}) = 0$$

(c) with coefficients

$$a_{11} = 0, \quad b_{11} = 0, \quad a_{22} = 1/\phi_\lambda, \quad b_{21} = 1, \quad b_{22} = \phi_\mu/\phi_\lambda$$

and the general solution

$$\lambda = x_1, \quad \Phi(\mu - x_2, x_3 + \phi) = 0$$

(d) with coefficients satisfying (3.6) and a general solution having two arbitrary functions of one argument.

Here $\Phi = \Phi(\xi_1,\ \xi_2)$, $\phi = \phi(\lambda,\ \mu)$ and $\psi = \psi(\mu)$ are arbitrary functions and $\Phi_{\xi_2} \neq 0$. In case (d) system (1.2) is said to be written in terms of Riemannian invariants. In [6] the condition that the system be written in terms of Riemannian invariants is required in the definition of a double wave.

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